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On Interpolation, Summation, and the Adjustment of Numerical Tables. By W. S. B. WOOLHOUSE, F.R.A.S.; F.I.A., &c.

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INTERPOLATION, generally considered, is the process of determining numerical values intermediate to a series of other values that are already known, and it is of extraordinary utility in many calculations, and especially in the formation of extensive tables. If, in the direct calculation of a table, the values to be tabulated depend upon a complicated function, difficult of calculation, considerable labour and corresponding risk of error may usually be avoided by computing the values for a certain succession of equidistant intervals, and afterwards determining by interpolation all other values that may be requisite to complete the table and adapt it to the practical purposes for which it is constructed. In such case the method is made use of as a powerful auxiliary at the option of the computer. If, however, the only available data are given values at certain regular intervals, and the mathematical form of the function is unknown, then the process of interpolation, under some of its modifications, becomes absolutely indispensable, and its true application is necessarily of great

importance where accuracy is required. The subject is closely allied to the theory of finite differences, but it is only incidentally introduced in mathematical treatises. It is therefore presumed that an independent discussion of the theory, with a proper regard to its practical bearings, and embracing new inquiries and a variety of original formulæ, might be useful as well as interesting to many persons engaged in calculations, and be specially acceptable also to the members of the Institute of Actuaries, since the most scientific and accurate methods of dealing with statistical facts, and of constructing and adjusting tables of mortality, are to be drawn from this theory.

A table of single entry usually consists of two columns. The first column contains what is technically called the "argument" of the table; that is, the number by which the table is entered in order to determine the value of the corresponding tabulated result shown in the second column. If the identical argument should appear in the table, the result is taken out by a simple inspection of the second column. But if the proposed argument should be intermediate to two contiguous arguments contained in the table, the required result, which is likewise intermediate, is determined by interpolation. The consecutive arguments of a table are in general sufficiently close to reduce this interpolation to the easiest kind, viz., that of simple proportion.

A table of double entry, or of double argument, is a tabular registry of results which depend on two arguments or two independent variables, such as a table exhibiting the numerical results of a certain combination of two lives. In like manner tables may be formed having more arguments than two, but they are necessarily cumbersome and unwieldy, and therefore very rarely resorted to.

Before entering upon the general mathematical investigation, we may adduce one or two elementary examples and illustrations of the principles which appertain to the differences of algebraic functions when the values of the independent variable proceed by equal gradations or equal intervals. To obtain as much simplicity as possible in the mathematical formulæ, one of these intervals is adopted as the unit of measurement with respect to the independent variable. The successive values of the function are usually placed in order in a vertical column. Their first differences are obtained by the subtraction of each value from that which immediately follows it, having always a due regard to the signs + and −. These differences are arranged in an adjoining column, each of them being placed between the corresponding values differenced.

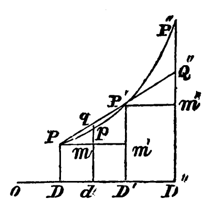
In precisely the same manner, by differencing this column of first differences, a column of second differences is obtained; by again differencing these last, a column of third differences is obtained, and so on.

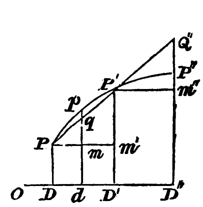
If the function be simply a given multiple of the variable added to a constant number—that is, an algebraic expression of the first degree—the first differences will be constant; and if the function be an algebraic expression of the second degree, the second differences will be constant. Thus, for example, we have the following differences:—

x	5		$3x$			x^2			$5+3x$			$5+3x+x^2$			$5+3x+x^2$		
1	5		3			1			8			9			9		
		0		+3			+3			+3			+6				
2	5		6		0	4		+2	11		0	15		+2		+14	
		0		3			5			3			8				
3	5		9		0	9		2	14		0	23		2	23		+8
		0		3			7			3			10				
4	5		12		0	16		2	17		0	33		2		22	
		0		3			9			3			12				
5	5		15		0	25		2	20		0	45		2	45		+8
		0		3			11			3			14				
6	5		18		0	36		+2	23		0	59		+2		+30	
		0		+3			+13			+3			+16				
7	5		21			49			26			75			75		

And, generally, if the function be of the nth degree, the nth order of differences will be constant, and those of the $n+1$th order will be zero.

The nature of interpolation, so far as first and second differences are concerned, admits of a tolerably clear geometrical illustration. In the annexed diagrams let the ordinate PD represent a function of the abscissa OD; and let PD, P′D′ be two consecutive values. Then, Pm' being parallel to DD′, P′m' will be the first difference. If the first difference be constant, or P′m' = Q″m'', the point P will continue to be in the given straight line PP′Q″, and qd= PD + qm will be an intermediate ordinate, of which the part PD is the primitive value, and the part qm, by similar triangles, is the simple proportional part of the first difference P′m'.





Now suppose a second difference to exist. If it be positive, the first difference P′m' will increase and have P″m'' in the *first* diagram, for its next succeeding value, and it is evident that the track of the point P will be a curve PP′P″ with its convex side turned downwards; and the

true intermediate value pd will be less than the value qd determined by a simple proportion of the first difference, or the correction on account of second difference will be subtractive.

In like manner if the second difference be negative, the first difference $P'm'$ will decrease, and the curve $PP'P''$ described by P will be, as in the *second* diagram, having its convex side turned upwards. Also the true intermediate value pd will evidently be greater than that of qd resulting from simple proportion, so that the correction on account of second difference will here be additive.

Hence it appears that in either case the correction for second difference has a sign contrary to that of the difference. In both diagrams $P''Q''$ represents the second difference, and pq the correction on its account. The greatest value of the correction pq occurs at the middle of the interval, when it becomes equal to one-eighth of the second difference. If, therefore, in any table it be found that the second difference is less than 8, it may generally be concluded that interpolation by simple proportion will be always true within a fractional part of an unit.

To proceed with the general subject, suppose a series of values $V_{-4} \ldots . V_{+4}$, of which the central value is V_0, to be successively differenced as far as the eighth order, and let the differences be designated as in the following scheme, in which

$$
\begin{aligned}
(a_0) &= \tfrac{1}{2}(a_{-1} + a_{+1}) \\
(c_0) &= \tfrac{1}{2}(c_{-1} + c_{+1}) \\
(e_0) &= \tfrac{1}{2}(e_{-1} + e_{+1}) \\
(g_0) &= \tfrac{1}{2}(g_{-1} + g_{+1});
\end{aligned}
$$

these arithmetical means being supposed to be inserted after the differencing is completed:—

$$
\begin{array}{l|l|l|l|l|l|l|l|l}
V_{-4} & & & & & & & & \\
 & a_{-4} & & & & & & & \\
V_{-3} & & b_{-3} & & & & & & \\
 & a_{-3} & & c_{-3} & & & & & \\
V_{-2} & & b_{-2} & & d_{-2} & & & & \\
 & a_{-2} & & c_{-2} & & e_{-2} & & & \\
V_{-1} & & b_{-1} & & d_{-1} & & f_{-1} & & \\
 & a_{-1} & & c_{-1} & & e_{-1} & & g_{-1} & \\
V_0 & (a_0) & b_0 & (c_0) & d_0 & (e_0) & f_0 & (g_0) & h_0 \\
 & a_{+1} & & c_{+1} & & e_{+1} & & g_{+1} & \\
V_{+1} & & b_{+1} & & d_{+1} & & f_{+1} & & \\
 & a_{+2} & & c_{+2} & & e_{+2} & & & \\
V_{+2} & & b_{+2} & & d_{+2} & & & & \\
 & a_{+3} & & c_{+3} & & & & & \\
V_{+3} & & b_{+3} & & & & & & \\
 & a_{+4} & & & & & & & \\
V_{+4} & & & & & & & &
\end{array}
$$

Moreover, let the form of the function, when developed in powers of x, be assumed to be

$$V_x = V_0 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + Hx^8 \ldots \text{(1)},$$

in which A, B, C, &c. denote constant coefficients.

Then, by substituting -4, -3, -2, $-1, 0, +1$, $+2$, $+3$, $+4$, successively for x, the following values are obtained :—

$$\begin{aligned}
V_{-4} &= V_0 - 4A + 16B - 64C + 256D - 1024E + 4096F - 16384G + 65536H \\
V_{-3} &= V_0 - 3A + 9B - 27C + 81D - 243E + 729F - 2187G + 6561H \\
V_{-2} &= V_0 - 2A + 4B - 8C + 16D - 32E + 64F - 128G + 256H \\
V_{-1} &= V_0 - A + B - C + D - E + F - G + H \\
V_0 &= V_0 \\
V_{+1} &= V_0 + A + B + C + D + E + F + G + H \\
V_{+2} &= V_0 + 2A + 4B + 8C + 16D + 32E + 64F + 128G + 256H \\
V_{+3} &= V_0 + 3A + 9B + 27C + 81D + 243E + 729F + 2187G + 6561H \\
V_{+4} &= V_0 + 4A + 16B + 64C + 256D + 1024E + 4096F + 16384G + 65536H
\end{aligned}$$

Hence, by differencing these, we get

$$\begin{aligned}
a_{-4} &= A - 7B + 37C - 175D + 781E - 3367F + 14197G - 58975H \\
a_{-3} &= A - 5B + 19C - 65D + 211E - 665F + 2059G - 6305H \\
a_{-2} &= A - 3B + 7C - 15D + 31E - 63F + 127G - 255H \\
a_{-1} &= A - B + C - D + E - F + G - H \\
a_{+1} &= A + B + C + D + E + F + G + H \\
a_{+2} &= A + 3B + 7C + 15D + 31E + 63F + 127G + 255H \\
a_{+3} &= A + 5B + 19C + 65D + 211E + 665F + 2059G + 6305H \\
a_{+4} &= A + 7B + 37C + 175D + 781E + 3367F + 14197G + 58975H
\end{aligned}$$

Differencing again, and so continuing, in accordance with the scheme,

$$\begin{aligned}
b_{-3} &= 2B - 18C + 110D - 570E + 2702F - 12138G + 52670H \\
b_{-2} &= 2B - 12C + 50D - 180E + 602F - 1932G + 6050H \\
b_{-1} &= 2B - 6C + 14D - 30E + 62F - 126G + 254H \\
b_0 &= 2B \quad \ast + 2D \quad \ast + 2F \quad \ast + 2H \\
b_{+1} &= 2B + 6C + 14D + 30E + 62F + 126G + 254H \\
b_{+2} &= 2B + 12C + 50D + 180E + 602F + 1932G + 6050H \\
b_{+3} &= 2B + 18C + 110D + 570E + 2702F + 12138G + 52670H
\end{aligned}$$

$$\begin{aligned}
c_{-3} &= 6C - 60D + 390E - 2100F + 10206G - 46620H \\
c_{-2} &= 6C - 36D + 150E - 540F + 1806G - 5796H \\
c_{-1} &= 6C - 12D + 30E - 60F + 126G - 252H \\
c_{+1} &= 6C + 12D + 30E + 60F + 126G + 252H \\
c_{+2} &= 6C + 36D + 150E + 540F + 1806G + 5796H \\
c_{+3} &= 6C + 60D + 390E + 2100F + 10206G + 46620H
\end{aligned}$$

$$
\begin{aligned}
d_{-2}&=24D-240E+1560F-8400G+40824H\\
d_{-1}&=24D-120E+\ \ 480F-1680G+\ \ 5544H\\
d_{0}\ &=24D\qquad *\quad +\ \ 120F\qquad *\quad +\ \ \ 504H\\
d_{+1}&=24D+120E+\ \ 480F+1680G+\ \ 5544H\\
d_{+2}&=24D+240E+1560F+8400G+40824H
\end{aligned}
$$

$$
\begin{aligned}
e_{-2}&=120E-1080F+6720G-35280H\\
e_{-1}&=120E-\ \ 360F+1680G-\ \ 5040H\\
e_{+1}&=120E+\ \ 360F+1680G+\ \ 5040H\\
e_{+2}&=120E+1080F+6720G+35280H
\end{aligned}
$$

$$
\begin{aligned}
f_{-1}&=720F-5040G+30240H\\
f_{0}\ &=720F\qquad *\qquad +10080H\\
f_{+1}&=720F+5040G+30240H
\end{aligned}
$$

$$
\begin{aligned}
g_{-1}&=5040G-20160H\\
g_{+1}&=5040G+20160H
\end{aligned}
$$

$$h_0=40320H.$$

Hence, observing that $(a_0)=\frac{1}{2}(a_{-1}+a_{+1})$, $(c_0)=\frac{1}{2}(c_{-1}+c_{+1})$, &c., the central values are

$$
\begin{aligned}
(a_0)&=A+C+E+G\\
b_0\ &=2(B+D+F+H)\\
(c_0)&=2.3(C+5E+21G)\\
d_0\ &=2.3.4(D+5F+21H)\\
(e_0)&=2.3.4.5(E+14G)\\
f_0\ &=2.3.4.5.6(F+14H)\\
(g_0)&=2.3.4.5.6.7G\\
h_0\ &=2.3.4.5.6.7.8H\quad .\quad .\quad .\ (2);
\end{aligned}
$$

and from these we obtain the following values of the coefficients :—

$$
\begin{aligned}
A&=(a_0)-\frac{(c_0)}{2.3}+\frac{(e_0)}{2.3.5}-\frac{(g_0)}{4.5.7}\\
B&=\frac{b_0}{2}-\frac{d_0}{2.3.4}+\frac{f_0}{2.3.5.6}-\frac{h_0}{4.5.7.8}\\
C&=\frac{(c_0)}{2.3}-\frac{(e_0)}{2.3.4}+\frac{7(g_0)}{2.3.4.5.6}\\
D&=\frac{d_0}{2.3.4}-\frac{f_0}{2.3.4.6}+\frac{7h_0}{2.3.4.5.6.8}\\
E&=\frac{(e_0)}{2.3.4.5}-\frac{14(g_0)}{2\ldots7}
\end{aligned}
$$

$$F=\frac{f_0}{2\ldots6}-\frac{14h_0}{2\ldots8}$$

$$G=\frac{(g_0)}{2\ldots7}$$

$$H=\frac{h_0}{2\ldots8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Let n denote the number of subdivisions contained in the interval. Then, if $\frac{x}{n}$ be substituted for x in the function (1), it will, for $x=1, 2, 3$, &c., represent the interpolated in place of the primitive values. Hence if accentuated symbols be used to denote values which appertain to the interpolated quantities and their differences, in reference to the same epoch V_0, we shall have

$$A'=\frac{1}{n}A,\ B'=\frac{1}{n^2}B,\ C'=\frac{1}{n^3}C, \text{ \&c.}$$

$$\therefore (a'_0)=A'+C'+E'+G'$$
$$=\frac{(a_0)}{n}-\frac{n^2-1}{2.3}\cdot\frac{(c_0)}{n^3}+\frac{(n^2-1)(4n^2-1)}{2.3.4.5}\cdot\frac{(e_0)}{n^5}$$
$$-\frac{(n^2-1)(4n^2-1)(9n^2-1)}{2\ldots.7}\cdot\frac{(g_0)}{n^7}$$

$$b'_0=2(B'+D'+F'+H')$$
$$=\frac{b_0}{n^2}-\frac{n^2-1}{3.4}\cdot\frac{d_0}{n^4}+\frac{(n^2-1)(4n^2-1)}{3.4.5.6}\cdot\frac{f_0}{n^6}$$
$$-\frac{(n^2-1)(4n^2-1)(9n^2-1)}{3\ldots.8}\cdot\frac{h_0}{n^8}$$

$$(c'_0)=2.3(C'+5E'+21G')$$
$$=\frac{(c_0)}{n^3}-\frac{n^2-1}{4}\cdot\frac{(e_0)}{n^5}+\frac{(n^2-1)(7n^2-3)}{4.5.6}\cdot\frac{(g_0)}{n^7}$$

$$d'_0=2.3.4(D'+5F'+21H')$$
$$=\frac{d_0}{n^4}-\frac{n^2-1}{6}\cdot\frac{f_0}{n^6}+\frac{(n^2-1)(7n^2-3)}{5.6.8}\cdot\frac{h_0}{n^8}$$

$$(e'_0)=2.3.4.5(E'+14G')=\frac{(e_0)}{n^5}-\frac{n^2-1}{3}\cdot\frac{(g_0)}{n^7}$$

$$f'_0=2\ldots.6(F'+14H')=\frac{f_0}{n^6}-\frac{n^2-1}{4}\cdot\frac{h_0}{n^8}$$

$$(g'_0)=2\ldots.7G'=\frac{(g_0)}{n^7}$$

$$h'_0=2\ldots.8H'=\frac{h_0}{n^8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (p)$$

By substituting the values of A, B, C, &c., given by (3), in the function (1), we get

$$
\begin{aligned}
V_x = V_0 &+ \left(a_0 - \frac{c_0}{2.3} + \frac{e_0}{2.3.5} - \frac{g_0}{4.5.7}\right)x \\
&+ \left(\frac{b_0}{2} - \frac{d_0}{2.3.4} + \frac{f_0}{2.3.5.6} - \frac{h_0}{4.5.7.8}\right)x^2 \\
&+ \left(\frac{c_0}{2.3} - \frac{e_0}{2.3.4} + \frac{7g_0}{2.3.4.5.6}\right)x^3 \\
&+ \left(\frac{d_0}{2.3.4} - \frac{f_0}{2.3.4.6} + \frac{7h_0}{2.3.4.5.6.8}\right)x^4 \\
&+ \left(\frac{e_0}{2.3.4.5} - \frac{g_0}{3.4.5.6}\right)x^5 \\
&+ \left(\frac{f_0}{2.3.4.5.6} - \frac{h_0}{3.4.5.6.8}\right)x^6 \\
&+ \frac{g_0}{2\ldots.7}x^7 + \frac{h_0}{2\ldots.8}x^8;
\end{aligned}
$$

that is,

$$
\begin{aligned}
V_x = V_0 + x\left(a_0 + \frac{x}{2}b_0\right) &+ \frac{x(x^2-1)}{2.3}\left(c_0 + \frac{x}{4}d_0\right) \\
&+ \frac{x(x^2-1)(x^2-4)}{2.3.4.5}\left(e_0 + \frac{x}{6}f_0\right) + \frac{x(x^2-1)(x^2-4)(x^2-9)}{2.3.4.5.6.7}\left(g_0 + \frac{x}{8}h_0\right) \ldots (\alpha)
\end{aligned}
$$

Substitute $a_0 = a_1 - \frac{1}{2}b_0$, $c_0 = c_1 - \frac{1}{2}d_0$, $e_0 = e_1 - \frac{1}{2}f_0$, $g_0 = g_1 - \frac{1}{2}h_0$, and

$$
\begin{aligned}
V_x = V_0 + x\left(a_1 + \frac{x-1}{2}b_0\right) &+ \frac{x(x^2-1)}{2.3}\left(c_1 + \frac{x-2}{4}d_0\right) \\
&+ \frac{x(x^2-1)(x^2-4)}{2.3.4.5}\left(e_1 + \frac{x-3}{6}f_0\right) \\
&+ \frac{x(x^2-1)(x^2-4)(x^2-9)}{2.3.4.5.6.7}\cdot\left(g_1 + \frac{x-4}{8}h_0\right) \ldots\ldots (\beta).
\end{aligned}
$$

The value of V_x, between V_0 and V_1, answering to any proposed fractional value of x, may be found by either of the formulæ (α), (β). The second formula (β) is the best adapted for calculation, and an isolated value may be conveniently deduced, without tables, by successively correcting the differences in a retrograde order, according to the following system of equations:—

$$g'' = g_1 - \frac{4-x}{8}h_0$$

$$f'' = f_0 + \frac{3+x}{7}g''$$

$$e''=e_1-\frac{3-x}{6}f''$$

$$d''=d_0+\frac{2+x}{5}e''$$

$$c''=c_1-\frac{2-x}{4}d''$$

$$b''=b_0+\frac{1+x}{3}c''$$

$$a''=a_1-\frac{1-x}{2}b''$$

$$V_x=V_0+xa''.$$

Interpolations are, however, seldom carried beyond the fourth order of differences, so that only the last four of these equations will be required in practice, the mean fourth difference $(d)=\frac{1}{2}(d_0+d_1)$ being then substituted for d''. If the calculation is only required to include the second order of differences, the last two equations only will be needed, and the mean second difference $(b)=\frac{1}{2}(b_0+b_1)$ will then take the place of b''. The formula (β), stopping at the fourth order of differences, is

$$V_x=V_0+xa_1-\frac{x(1-x)}{2}b_0-\frac{x(1-x^2)}{2.3}c_1+\frac{x(1-x^2)(2-x)}{2.3.4}(d),$$

and it is from this formula that the tables have been constructed. The corresponding system of corrections for an independent calculation without tables are

$$c''=c_1-\frac{2-x}{4}(d)$$

$$b''=b_0+\frac{1+x}{3}c''$$

$$a''=a_1-\frac{1-x}{2}b''$$

$$V_x=V_0+xa''.$$

If it were required to bisect the interval, or to interpose the middle value between V_0 and V_1, then $x=\frac{1}{2}$, and

$$c''=c_1-\tfrac{3}{8}(d)$$

$$b''=b_0+\tfrac{1}{2}c''$$

$$a''=a_1-\tfrac{1}{4}b''$$

$$V_{\frac{1}{2}}=V_0+\tfrac{1}{2}a'',$$

which are easy of calculation.

To divide the interval into three, or to interpose two values between V_0 and V_1, x must take the values $\frac{1}{3}$, $\frac{2}{3}$.

To divide the interval into four, or to interpose three values, x must take the values $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$.

To divide the interval into five, x must take the values $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$.

And, generally, to divide the interval into n parts, or to interpose $n-1$ values between V_0 and V_1, x must take the values $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \frac{n-1}{n}$.

The term in the formula, which depends upon a difference of any order, is commonly called the "equation" of that difference. Thus, for example, the calculated value of the fourth term, involving c_1, would be called the "equation of the third difference."

Preparatory to the construction of tables for the intervals in general use—viz., five and ten—the values of the several coefficients, according to the formula (β), were first calculated and differenced. They are as follows :—

Interval 5.

x.	Coeff. of b_0 $-\frac{x(1-x)}{2}$.			Coeff. of c_1 $-\frac{x(1-x^2)}{2.3}$.			Coeff. of (d) $\frac{x(1-x^2)(2-x)}{2.3.4}$		
·0	−·00			−·000			·0000		
		−8			−32			+144	
·2	·08		+4	·032		+8	·0144		−64
		−4			24			80	
·4	·12		4	·056		16	·0224		80
		0			−8			0	
·6	·12		4	·064		24	·0224		80
		+4			+16			−80	
·8	·08		+4	·048		+32	·0144		−64
		+8			+48			−144	
1·0	−·00			−·000			·0000		

Interval 10.

x.	Coeff. of b_0.			Coeff. of c_1.			Coeff. of (d).		
·0	−·000			−·0000			·0000000		
		−45			−165			+78375	
·1	·045		+10	·0165		+10	·0078375		−12750
		35			155			65625	
·2	·080		10	·0320		20	·0144000		16250
		25			135			49375	
·3	·105		10	·0455		30	·0193375		18750
		15			105			30625	
·4	·120		10	·0560		40	·0224000		20250
		−5			65			+10375	
·5	·125		10	·0625		50	·0234375		20750
		+5			−15			−10375	
·6	·120		10	·0640		60	·0224000		20250
		15			+45			30625	
·7	·105		10	·0595		70	·0193375		18750
		25			115			49375	
·8	·080		10	·0480		80	·0144000		16250
		35			195			65625	
·9	·045		+10	·0285		+90	·0078375		−12750
		+45			+285			−78375	
1·0	−·000			−·0000			·0000000		

The tables, pages 74—82, have been formed from these coefficients so as to give the greatest possible facility in deducing in succession a complete set of interpolated values. They do not contain the equations of the differences. Beginning with $x=0$, the first tabulated column contains the leading first difference of the equations, and the other columns contain severally their second differences. But these differences have not been calculated with separate coefficients. The equations themselves were first determined to the nearest terminal digit, and the tabular differences then deduced from them. By this expedient the tables are much simplified. The employment of more than one additional place of figures becomes unnecessary, as there can be no accumulation of error, and a satisfactory check is provided by the accurate agreement of the final number with the succeeding value of the given series.*

The tabular numbers answering to consecutive arguments necessarily, and properly, exhibit frequent irregularities in their progressions, in consequence of their having been determined in the manner just described. When, however, they become combined together in the mechanical process of interpolation, the results, being correct in principle, ought in all cases to observe a due order of progression.

The order of succession of the several corrections contained in each table is indicated by corresponding numbers at the head of the respective columns.

The rule of signs to be observed, in taking out the corrections from the tables, is that

$$\left.\begin{matrix}+\\-\end{matrix}\right\}\text{ indicates a sign }\left\{\begin{matrix}\text{the same as}\\\text{contrary to}\end{matrix}\right\}\text{ that of the difference.}$$

Thus with the second and third differences all the corrections excepting the first have the same sign as that of the difference. With the fourth difference, however, this is reversed.

The following example, for an interval of 10, will serve to practically exemplify the use of the tables.

Example.—The logarithms of the numbers 220, 230, 240, 250, 260, 270, being given to seven places of decimals, let it be required to find the logarithms of all the numbers from 240 to 250.

* Mr. William Godward, Jun., of the *Nautical Almanac* Office, some years ago constructed some improved Astronomical Interpolation Tables, printed in 1857, in which he ingeniously introduced an expedient of a similar kind to that which is here adopted, and thereby essentially simplified the extensive work of interpolation carried on in that Office. The method of proceeding by continuous summation was originally suggested, many years ago, by Mr. Richard Farley, another gentleman in the same Office, who possesses a remarkable and instinctive talent for computation.

The given logarithms, differenced to the fourth order, are as follows :—

No.	Log.				
220	2·3424227				
		+193051			
230	2·3617278		−8217		
		184834		+671	
240	V_0.. 2·3802112		b_0... 7546		−80
		a_1... 177288		c ... 591	
250	2·3979400		6955		−64
		170333		+527	
260	2·4149733		−6428		−144
		+163905			
270	2·4313638				(d)... −72

And the whole of the work of interpolation will stand thus :—

Interval 10.

b_0 −7546	c_1 +591	(d) −72	$\frac{1}{10}a_1$ +17728·8	δ_1	V_0 2·3802112·0	= log 240
+339·6	− 9·8	− ·5	s +329·3	+18058·1	2·3820170·1	„ „ 241
− 75·5	+ ·7	+·0	δ_2 − 74·8	17983·3	838153·4	„ „ 242
*75·5	1·2	·2	*74·1	*17909·2	856062·6	„ „ 243
75·4	1·7	·1	73·6	17835·6	873898·2	„ „ 244
75·5	2·3	·1	73·1	17762·5	891660·7	„ „ 245
75·4	3·0	·2	72·2	17690·3	909351·0	„ „ 246
75·5	3·6	·1	71·8	17618·5	926969·5	„ „ 247
75·4	4·1	·1	71·2	17547·3	944516·8	„ „ 248
75·5	4·7	·2	70·6	17476·7	961993·5	„ „ 249
−75·5	+5·3	+·0	− 70·2	+17406·5	2·3979400·0	V_1 „ 250

Here the ten corrections taken from the respective tables, under 1, 2, 3 . . . 10, are put down in the first three columns. In taking out these numbers, only the nearest argument is used, and when the argument is large, any requisite integral multiple of the highest tabulated values may be included.

The fourth column of the calculation contains the sum of the three preceding columns taken horizontally, and at the head of this column is inserted the value of $\frac{1}{5}a_1$ for interval 5, or $\frac{1}{10}a_1$ for interval 10. The first of the sums in this column is denoted by s, and those which follow it are values of δ_2.

The fifth column contains values of δ_1. The first of these is found by adding s to the number above it; and the others are found by successively applying the several values of δ_2 contained in the preceding column.

At the head of the sixth column is inserted the value of V_0, from which the interpolation is to commence, and the required

* It may not be considered necessary to repeat the leading figures in these columns.

interpolated values in this column are finally obtained by successively applying the several values of δ_1. The whole of the work is in all cases ultimately checked by the exact agreement with V_1 of the last number in this column, and it will be found in the foregoing example that the several results are quite as accurate as the primitive values from which they have been computed.

It may be further remarked that the progression of the interpolated results is also exhibited in the calculation, since the values δ_1 and δ_2 are those of their first and second differences, so that the redifferencing of the final results is anticipated, and the regularity of the progressions is such that it is next to impossible that any accidental error could exist.

Where an asterisk (*) appears in a table the numbers are rigorously accurate, and may be incorporated *ad libitum*, by simple addition, if any extension of the table should be required.

NOTE.—If the coefficients A, B, C, &c., were deduced from the expressions for a_{+1}, b_{+1}, c_{+2}, d_{+2}, page 65, the resulting formula would be

$$V_x = V_0 + x a_1 + \frac{x(x-1)}{2} b_1 + \frac{x(x-1)(x-2)}{2.3} c_2 + \frac{x(x-1)(x-2)(x-3)}{2.3.4} d_2 + \&c.,$$

which is the ordinary development of a function according to the theory of finite differences. But this formula is less convergent and less accurate, for the purposes of interpolation, than the formula (β). It involves the leading differences, a_1, b_1, c_2, &c., which are derived exclusively from V_0, V_1, V_2, &c., without having any regard to the values which precede V_0. Whereas the formula (β) involves a system of differences, which occupy a central region and depend equally on the values which precede and follow the interpolated interval. This comparison may be conceived geometrically by considering the process of interpolation as analogous to the determination of the particular contour of a part of a curve, which connects two consecutive points, from a knowledge of the condition that the curve, when continued both ways, shall pass through a number of other given points both right and left. Supposing the portion of the curve to be drawn by hand, it would manifestly be more consistently and more accurately accomplished if the points on both sides of it were taken into account than if the attention were exclusively directed to the points posited on one side alone. In the latter case, in fact, the curve would be slightly different, according as the points on the right or the left should happen to be accepted as the guide for its determination; and a corresponding discrepancy would be found to exist in the results of actual calculation, if the primitive values were differenced in the reverse order V_1, V_0, . . . V_{-4}. But with the formula (β), on which the following tables have been constructed, the order in which the quantities are taken would be quite immaterial.

Interval 5, *Second Difference* b_0.

b_0	1	2 5	3 4	b_0	1	2 5	3 4
1	− ·1	+ ·1	·0	51	−4·1	+2·1	2·0
2	·2	·1	·1	52	4·2	2·1	2·1
3	·3	·2	·1	53	4·3	2·2	2·1
4	·3	·1	·2	54	4·3	2·1	2·2
*5	·4	·2	·2	55	4·4	2·2	2·2
6	− ·5	+ ·3	·2	56	−4·5	+2·3	2·2
7	·6	·3	·3	57	4·6	2·3	2·3
8	·7	·4	·3	58	4·7	2·4	2·3
9	·7	·3	·4	59	4·7	2·3	2·4
10	·8	·4	·4	60	4·8	2·4	2·4
11	− ·9	+ ·5	·4	61	−4·9	+2·5	2·4
12	1·0	·5	·5	62	5·0	2·5	2·5
13	1·1	·6	·5	63	5·1	2·6	2·5
14	1·1	·5	·6	64	5·1	2·5	2·6
15	1·2	·6	·6	65	5·2	2·6	2·6
16	−1·3	+ ·7	·6	66	−5·3	+2·7	2·6
17	1·4	·7	·7	67	5·4	2·7	2·7
18	1·5	·8	·7	68	5·5	2·8	2·7
19	1·5	·7	·8	69	5·5	2·7	2·8
20	1·6	·8	·8	70	5·6	2·8	2·8
21	−1·7	+ ·9	·8	71	−5·7	+2·9	2·8
22	1·8	·9	·9	72	5·8	2·9	2·9
23	1·9	1·0	·9	73	5·9	3·0	2·9
24	1·9	·9	1·0	74	5·9	2·9	3·0
25	2·0	1·0	1·0	75	6·0	3·0	3·0
26	−2·1	+1·1	1·0	76	−6·1	+3·1	3·0
27	2·2	1·1	1·1	77	6·2	3·1	3·1
28	2·3	1·2	1·1	78	6·3	3·2	3·1
29	2·3	1·1	1·2	79	6·3	3·1	3·2
30	2·4	1·2	1·2	80	6·4	3·2	3·2
31	−2·5	+1·3	1·2	81	−6·5	+3·3	3·2
32	2·6	1·3	1·3	82	6·6	3·3	3·3
33	2·7	1·4	1·3	83	6·7	3·4	3·3
34	2·7	1·3	1·4	84	6·7	3·3	3·4
35	2·8	1·4	1·4	85	6·8	3·4	3·4
36	−2·9	+1·5	1·4	86	−6·9	+3·5	3·4
37	3·0	1·5	1·5	87	7·0	3·5	3·5
38	3·1	1·6	1·5	88	7·1	3·6	3·5
39	3·1	1·5	1·6	89	7·1	3·5	3·6
40	3·2	1·6	1·6	90	7·2	3·6	3·6
41	−3·3	+1·7	1·6	91	−7·3	+3·7	3·6
42	3·4	1·7	1·7	92	7·4	3·7	3·7
43	3·5	1·8	1·7	93	7·5	3·8	3·7
44	3·5	1·7	1·8	94	7·5	3·7	3·8
45	3·6	1·8	1·8	95	7·6	3·8	3·8
46	−3·7	+1·9	1·8	96	−7·7	+3·9	3·8
47	3·8	1·9	1·9	97	7·8	3·9	3·9
48	3·9	2·0	1·9	98	7·9	4·0	3·9
49	3·9	1·9	2·0	99	7·9	3·9	4·0
50	−4·0	+2·0	2·0	100	−8·0	+4·0	4·0

Interval 5, *Third Difference* c_1.

c_1	1	2	3	4	5	c_1	1	2	3	4	5
5	– ·2	+ ·1	·1	·1	·1	255	– 8·2	+2·1	4·1	6·1	8·1
10	·3	·1	·1	·2	·4	260	8·3	2·1	4·1	6·2	8·4
15	·5	·2	·2	·3	·5	265	8·5	2·2	4·2	6·3	8·5
20	·6	·1	·3	·5	·7	270	8·6	2·1	4·3	6·5	8·7
*25	·8	·2	·4	·6	·8	275	8·8	2·2	4·4	6·6	8·8
30	–1·0	+ ·3	·5	·7	·9	280	– 9·0	+2·3	4·5	6·7	8·9
35	1·1	·3	·5	·8	1·2	285	9·1	2·3	4·5	6·8	9·2
40	1·3	·4	·6	·9	1·3	290	9·3	2·4	4·6	6·9	9·3
45	1·4	·3	·7	1·1	1·5	295	9·4	2·3	4·7	7·1	9·5
50	1·6	·4	·8	1·2	1·6	300	9·6	2·4	4·8	7·2	9·6
55	–1·8	+ ·5	·9	1·3	1·7	305	– 9·8	+2·5	4·9	7·3	9·7
60	1·9	·5	·9	1·4	2·0	310	9·9	2·5	4·9	7·4	10·0
65	2·1	·6	1·0	1·5	2·1	315	10·1	2·6	5·0	7·5	10·1
70	2·2	·5	1·1	1·7	2·3	320	10·2	2·5	5·1	7·7	10·3
75	2·4	·6	1·2	1·8	2·4	325	10·4	2·6	5·2	7·8	10·4
80	–2·6	+ ·7	1·3	1·9	2·5	330	–10·6	+2·7	5·3	7·9	10·5
85	2·7	·7	1·3	2·0	2·8	335	10·7	2·7	5·3	8·0	10·8
90	2·9	·8	1·4	2·1	2·9	340	10·9	2·8	5·4	8·1	10·9
95	3·0	·7	1·5	2·3	3·1	345	11·0	2·7	5·5	8·3	11·1
100	3·2	·8	1·6	2·4	3·2	350	11·2	2·8	5·6	8·4	11·2
105	–3·4	+ ·9	1·7	2·5	3·3	355	–11·4	+2·9	5·7	8·5	11·3
110	3·5	·9	1·7	2·6	3·6	360	11·5	2·9	5·7	8·6	11·6
115	3·7	1·0	1·8	2·7	3·7	365	11·7	3·0	5·8	8·7	11·7
120	3·8	·9	1·9	2·9	3·9	370	11·8	2·9	5·9	8·9	11·9
125	4·0	1·0	2·0	3·0	4·0	375	12·0	3·0	6·0	9·0	12·0
130	–4·2	+1·1	2·1	3·1	4·1	380	–12·2	+3·1	6·1	9·1	12·1
135	4·3	1·1	2·1	3·2	4·4	385	12·3	3·1	6·1	9·2	12·4
140	4·5	1·2	2·2	3·3	4·5	390	12·5	3·2	6·2	9·3	12·5
145	4·6	1·1	2·3	3·5	4·7	395	12·6	3·1	6·3	9·5	12·7
150	4·8	1·2	2·4	3·6	4·8	400	12·8	3·2	6·4	9·6	12·8
155	–5·0	+1·3	2·5	3·7	4·9	405	–13·0	+3·3	6·5	9·7	12·9
160	5·1	1·3	2·5	3·8	5·2	410	13·1	3·3	6·5	9·8	13·2
165	5·3	1·4	2·6	3·9	5·3	415	13·3	3·4	6·6	9·9	13·3
170	5·4	1·3	2·7	4·1	5·5	420	13·4	3·3	6·7	10·1	13·5
175	5·6	1·4	2·8	4·2	5·6	425	13·6	3·4	6·8	10·2	13·6
180	–5·8	+1·5	2·9	4·3	5·7	430	–13·8	+3·5	6·9	10·3	13·7
185	5·9	1·5	2·9	4·4	6·0	435	13·9	3·5	6·9	10·4	14·0
190	6·1	1·6	3·0	4·5	6·1	440	14·1	3·6	7·0	10·5	14·1
195	6·2	1·5	3·1	4·7	6·3	445	14·2	3·5	7·1	10·7	14·3
200	6·4	1·6	3·2	4·8	6·4	450	14·4	3·6	7·2	10·8	14·4
205	–6·6	+1·7	3·3	4·9	6·5	455	–14·6	+3·7	7·3	10·9	14·5
210	6·7	1·7	3·3	5·0	6·8	460	14·7	3·7	7·3	11·0	14·8
215	6·9	1·8	3·4	5·1	6·9	465	14·9	3·8	7·4	11·1	14·9
220	7·0	1·7	3·5	5·3	7·1	470	15·0	3·7	7·5	11·3	15·1
225	7·2	1·8	3·6	5·4	7·2	475	15·2	3·8	7·6	11·4	15·2
230	–7·4	+1·9	3·7	5·5	7·3	480	–15·4	+3·9	7·7	11·5	15·3
235	7·5	1·9	3·7	5·6	7·6	485	15·5	3·9	7·7	11·6	15·6
240	7·7	2·0	3·8	5·7	7·7	490	15·7	4·0	7·8	11·7	15·7
245	7·8	1·9	3·9	5·9	7·9	495	15·8	3·9	7·9	11·9	15·9
250	– 8·0	+ 2·0	4·0	6·0	8·0	500	–16·0	+ 4·0	8·0	12·0	16·0

Interval 5, *Mean Fourth Difference* (*d*).

(*d*)	1	2 5	3 4	(*d*)	1	2 5	3 4
5	+ ·1	− ·1	·0	255	+3·7	−1·7	2·0
10	·1	·0	·1	260	3·7	1·6	2·1
15	·2	·1	·1	265	3·8	1·7	2·1
20	·3	·2	·1	270	3·9	1·8	2·1
25	·4	·2	·2	275	4·0	1·8	2·2
30	+ ·4	− ·1	·3	280	+4·0	−1·7	2·3
35	·5	·2	·3	285	4·1	1·8	2·3
40	·6	·3	·3	290	4·2	1·9	2·3
45	·6	·2	·4	295	4·2	1·8	2·4
50	·7	·3	·4	300	4·3	1·9	2·4
55	+ ·8	− ·4	·4	305	+4·4	−2·0	2·4
60	·9	·5	·4	310	4·5	2·1	2·4
65	·9	·3	·6	315	4·5	1·9	2·6
70	1·0	·4	·6	320	4·6	2·0	2·6
75	1·1	·5	·6	325	4·7	2·1	2·6
80	+1·2	− ·6	·6	330	+4·8	−2·2	2·6
85	1·2	·5	·7	335	4·8	2·1	2·7
90	1·3	·6	·7	340	4·9	2·2	2·7
95	1·4	·7	·7	345	5·0	2·3	2·7
100	1·4	·6	·8	350	5·0	2·2	2·8
105	+1·5	− ·6	·9	355	+5·1	−2·2	2·9
110	1·6	·7	·9	360	5·2	2·3	2·9
115	1·7	·8	·9	365	5·3	2·4	2·9
120	1·7	·7	1·0	370	5·3	2·3	3·0
*125	1·8	·8	1·0	375	5·4	2·4	3·0
130	+1·9	− ·9	1·0	380	+5·5	−2·5	3·0
135	1·9	·8	1·1	385	5·5	2·4	3·1
140	2·0	·9	1·1	390	5·6	2·5	3·1
145	2·1	1·0	1·1	395	5·7	2·6	3·1
150	2·2	1·0	1·2	400	5·8	2·6	3·2
155	+2·2	− ·9	1·3	405	+5·8	−2·5	3·3
160	2·3	1·0	1·3	410	5·9	2·6	3·3
165	2·4	1·1	1·3	415	6·0	2·7	3·3
170	2·4	1·0	1·4	420	6·0	2·6	3·4
175	2·5	1·1	1·4	425	6·1	2·7	3·4
180	+2·6	−1·2	1·4	430	+6·2	−2·8	3·4
185	2·7	1·3	1·4	435	6·3	2·9	3·4
190	2·7	1·1	1·6	440	6·3	2·7	3·6
195	2·8	1·2	1·6	445	6·4	2·8	3·6
200	2·9	1·3	1·6	450	6·5	2·9	3·6
205	+3·0	−1·4	1·6	455	+6·6	−3·0	3·6
210	3·0	1·3	1·7	460	6·6	2·9	3·7
215	3·1	1·4	1·7	465	6·7	3·0	3·7
220	3·2	1·5	1·7	470	6·8	3·1	3·7
225	3·2	1·4	1·8	475	6·8	3·0	3·8
230	+3·3	−1·4	1·9	480	+6·9	−3·0	3·9
235	3·4	1·5	1·9	485	7·0	3·1	3·9
240	3·5	1·6	1·9	490	7·1	3·2	3·9
245	3·5	1·5	2·0	495	7·1	3·1	4·0
250	+3·6	−1·6	2·0	500	+7·2	−3·2	4·0

Interval 10, *Second Difference* b_0.

b_0	1	2 10	3 9	4 8	5 7	6	b_0	1	2 10	3 9	4 8	5 7	6
1	− ·1	+·1	·0	·0	·0	·0	51	−2·3	+ ·5	·6	·4	·5	·6
2	·1	·0	·1	·0	·0	·0	52	2·3	·5	·5	·5	·5	·6
3	·1	·0	·0	·0	·1	·0	53	2·4	·6	·5	·5	·5	·6
4	·2	·1	·0	·0	·1	·0	54	2·4	·5	·5	·6	·5	·6
5	·2	·0	·1	·0	·1	·0	55	2·5	·6	·5	·6	·5	·6
6	− ·3	+·1	·1	·0	·1	·0	56	−2·5	+ ·5	·6	·6	·5	·6
7	·3	·1	·0	·1	·0	·2	57	2·6	·6	·6	·6	·5	·6
8	·3	·0	·0	·2	·1	·0	58	2·6	·6	·5	·6	·6	·6
9	·4	·1	·1	·0	·1	·2	59	2·6	·5	·6	·6	·6	·6
10	·5	·2	·1	·0	·1	·2	60	2·7	·6	·6	·6	·6	·6
11	− ·5	+·1	·2	·0	·1	·2	61	−2·8	+ ·7	·6	·6	·6	·6
12	·5	·1	·1	·1	·1	·2	62	2·8	·6	·7	·6	·6	·6
13	·6	·2	·1	·1	·1	·2	63	2·8	·6	·6	·6	·7	·6
14	·6	·1	·1	·2	·1	·2	64	2·9	·7	·6	·6	·7	·6
15	·7	·2	·1	·2	·1	·2	65	2·9	·6	·7	·6	·7	·6
16	− ·7	+·1	·2	·2	·1	·2	66	−3·0	+ ·7	·7	·6	·7	·6
17	·8	·2	·2	·2	·1	·2	67	3·0	·7	·6	·7	·6	·8
18	·8	·2	·1	·2	·2	·2	68	3·0	·6	·6	·8	·7	·6
19	·8	·1	·2	·2	·2	·2	69	3·1	·7	·7	·6	·7	·8
*20	·9	·2	·2	·2	·2	·2	70	3·2	·8	·7	·6	·7	·8
21	−1·0	+·3	·2	·2	·2	·2	71	−3·2	+·7	·8	·6	·7	·8
22	1·0	·2	·3	·2	·2	·2	72	3·2	·7	·7	·7	·7	·8
23	1·0	·2	·2	·2	·3	·2	73	3·3	·8	·7	·7	·7	·8
24	1·1	·3	·2	·2	·3	·2	74	3·3	·7	·7	·8	·7	·8
25	1·1	·2	·3	·2	·3	·2	75	3·4	·8	·7	·8	·7	·8
26	−1·2	+·3	·3	·2	·3	·2	76	−3·4	+·7	·8	·8	·7	·8
27	1·2	·3	·2	·3	·2	·4	77	3·5	·8	·8	·8	·7	·8
28	1·2	·2	·2	·4	·3	·2	78	3·5	·8	·7	·8	·8	·8
29	1·3	·3	·3	·2	·3	·4	79	3·5	·7	·8	·8	·8	·8
30	1·4	·4	·3	·2	·3	·4	80	3·6	·8	·8	·8	·8	·8
31	−1·4	+·3	·4	·2	·3	·4	81	−3·7	+·9	·8	·8	·8	·8
32	1·4	·3	·3	·3	·3	·4	82	3·7	·8	·9	·8	·8	·8
33	1·5	·4	·3	·3	·3	·4	83	3·7	·8	·8	·8	·9	·8
34	1·5	·3	·3	·4	·3	·4	84	3·8	·9	·8	·8	·9	·8
35	1·6	·4	·3	·4	·3	·4	85	3·8	·8	·9	·8	·9	·8
36	−1·6	+·3	·4	·4	·3	·4	86	−3·9	+·9	·9	·8	·9	·8
37	1·7	·4	·4	·4	·3	·4	87	3·9	·9	·8	·9	·8	1·0
38	1·7	·4	·3	·4	·4	·4	88	3·9	·8	·8	1·0	·9	·8
39	1·7	·3	·4	·4	·4	·4	89	4·0	·9	·9	·8	·9	1·0
40	1·8	·4	·4	·4	·4	·4	90	4·1	1·0	·9	·8	·9	1·0
41	−1·9	+·5	·4	·4	·4	·4	91	−4·1	+ ·9	1·0	·8	·9	1·0
42	1·9	·4	·5	·4	·4	·4	92	4·1	·9	·9	·9	·9	1·0
43	1·9	·4	·4	·4	·5	·4	93	4·2	1·0	·9	·9	·9	1·0
44	2·0	·5	·4	·4	·5	·4	94	4·2	·9	·9	1·0	·9	1·0
45	2·0	·4	·5	·4	·5	·4	95	4·3	1·0	·9	1·0	·9	1·0
46	−2·1	+·5	·5	·4	·5	·4	96	−4·3	+ ·9	1·0	1·0	·9	1·0
47	2·1	·5	·4	·5	·4	·6	97	4·4	1·0	1·0	1·0	·9	1·0
48	2·1	·4	·4	·6	·5	·4	98	4·4	1·0	·9	1·0	1·0	1·0
49	2·2	·5	·5	·4	·5	·6	99	4·4	·9	1·0	1·0	1·0	1·0
50	−2·3	+·6	·5	·4	·5	·6	100	−4·5	+1·0	1·0	1·0	1·0	1·0

Interval 10, *Third Difference* c_1.

c_1	1	2	3	4	5	6	7	8	9	10
5	− ·1	+ ·0	·0	·1	·0	·0	·0	·1	·0	·0
10	·2	·1	·0	·0	·0	·1	·0	·1	·1	·1
15	·3	·1	·0	·1	·0	·0	·2	·1	·1	·1
20	·3	·0	·0	·1	·1	·0	·2	·1	·2	·2
25	·4	·0	·1	·0	·1	·2	·1	·2	·2	·2
30	− ·5	+ ·0	·1	·1	·1	·2	·1	·2	·3	·3
35	·6	·1	·0	·1	·2	·2	·1	·3	·3	·3
40	·7	·1	·1	·1	·1	·2	·3	·3	·3	·3
45	·7	·0	·1	·1	·2	·2	·3	·3	·4	·4
50	·8	·0	·1	·2	·2	·2	·3	·4	·4	·4
55	− ·9	+ ·0	·2	·1	·3	·2	·3	·4	·5	·5
60	1·0	·1	·1	·2	·2	·3	·3	·5	·5	·5
65	1·1	·1	·1	·2	·3	·3	·4	·5	·5	·5
70	1·2	·1	·2	·2	·2	·4	·4	·5	·6	·6
75	1·2	·0	·2	·2	·3	·4	·4	·6	·6	·6
80	− 1·3	+ ·1	·1	·2	·4	·4	·4	·6	·7	·7
85	1·4	·1	·1	·3	·4	·4	·4	·7	·7	·7
90	1·5	·1	·2	·3	·3	·4	·6	·7	·7	·7
95	1·6	·1	·3	·2	·4	·4	·6	·7	·8	·8
100	1·6	·0	·2	·4	·4	·4	·6	·8	·8	·8
105	− 1·7	+ ·0	·3	·3	·4	·6	·6	·7	·8	1·0
110	1·8	·1	·2	·3	·5	·6	·6	·8	·8	1·0
115	1·9	·1	·3	·3	·4	·6	·8	·7	·9	1·1
120	2·0	·2	·2	·3	·5	·6	·8	·8	·9	1·1
125	2·1	·2	·2	·4	·5	·6	·8	·8	1·0	1·2
130	− 2·2	+ ·2	·3	·3	·6	·6	·8	·9	1·0	1·2
135	2·2	·1	·3	·4	·5	·7	·8	·9	1·1	1·3
140	2·3	·1	·3	·4	·6	·7	·9	·9	1·1	1·3
145	2·4	·2	·2	·5	·5	·8	·9	1·0	1·1	1·3
150	2·5	·2	·3	·4	·6	·8	·9	1·0	1·2	1·4
155	− 2·6	+ ·2	·3	·5	·6	·8	·9	1·1	1·2	1·4
160	2·6	·1	·3	·5	·7	·8	·9	1·1	1·3	1·5
165	2·7	·1	·4	·5	·6	·8	1·1	1·1	1·3	1·5
170	2·8	·2	·3	·5	·7	·8	1·1	1·2	1·3	1·5
175	2·9	·2	·3	·6	·7	·8	1·1	1·2	1·4	1·6
180	− 3·0	+ ·2	·4	·5	·7	1·0	1·0	1·3	1·4	1·6
185	3·1	·3	·3	·5	·8	1·0	1·0	1·3	1·5	1·7
190	3·2	·3	·4	·5	·7	1·0	1·2	1·3	1·5	1·7
195	3·2	·2	·3	·7	·7	1·0	1·2	1·3	1·6	1·8
*200	3·3	·2	·4	·6	·8	1·0	1·2	1·4	1·6	1·8
205	− 3·4	+ ·2	·4	·7	·8	1·0	1·2	1·5	1·6	1·8
210	3·5	·3	·4	·6	·8	1·1	1·2	1·5	1·7	1·9
215	3·6	·3	·4	·7	·8	1·0	1·4	1·5	1·7	1·9
220	3·6	·2	·4	·7	·9	1·0	1·4	1·5	1·8	2·0
225	3·7	·2	·5	·6	·9	1·2	1·3	1·6	1·8	2·0
230	− 3·8	+ ·2	·5	·7	·9	1·2	1·3	1·6	1·9	2·1
235	3·9	·3	·4	·7	1·0	1·2	1·3	1·7	1·9	2·1
240	4·0	·3	·5	·7	·9	1·2	1·5	1·7	1·9	2·1
245	4·0	·2	·5	·7	1·0	1·2	1·5	1·7	2·0	2·2
250	− 4·1	+ ·2	·5	·8	1·0	1·2	1·5	1·8	2·0	2·2

Interval 10, *Third Difference* c_1 (continued).

c_1	1	2	3	4	5	6	7	8	9	10
255	−4·2	+·2	·6	·7	1·1	1·2	1·5	1·8	2·1	2·3
260	4·3	·3	·5	·8	1·0	1·3	1·5	1·9	2·1	2·3
265	4·4	·3	·5	·8	1·1	1·3	1·6	1·9	2·1	2·3
270	4·5	·3	·6	·8	1·0	1·4	1·6	1·9	2·2	2·4
275	4·5	·2	·6	·8	1·1	1·4	1·6	2·0	2·2	2·4
280	−4·6	+·3	·5	·8	1·2	1·4	1·6	2·0	2·3	2·5
285	4·7	·3	·5	·9	1·2	1·4	1·6	2·1	2·3	2·5
290	4·8	·3	·6	·9	1·1	1·4	1·8	2·1	2·3	2·5
295	4·9	·3	·7	·8	1·2	1·4	1·8	2·1	2·4	2·6
300	4·9	·2	·6	1·0	1·2	1·4	1·8	2·2	2·4	2·6
305	−5·0	+·2	·7	·9	1·2	1·6	1·8	2·1	2·4	2·8
310	5·1	·3	·6	·9	1·3	1·6	1·8	2·2	2·4	2·8
315	5·2	·3	·7	·9	1·2	1·6	2·0	2·1	2·5	2·9
320	5·3	·4	·6	·9	1·3	1·6	2·0	2·2	2·5	2·9
325	5·4	·4	·6	1·0	1·3	1·6	2·0	2·2	2·6	3·0
330	−5·5	+·4	·7	·9	1·4	1·6	2·0	2·3	2·6	3·0
335	5·5	·3	·7	1·0	1·3	1·7	2·0	2·3	2·7	3·1
340	5·6	·3	·7	1·0	1·4	1·7	2·1	2·3	2·7	3·1
345	5·7	·4	·6	1·1	1·3	1·8	2·1	2·4	2·7	3·1
350	5·8	·4	·7	1·0	1·4	1·8	2·1	2·4	2·8	3·2
355	−5·9	+·4	·7	1·1	1·4	1·8	2·1	2·5	2·8	3·2
360	5·9	·3	·7	1·1	1·5	1·8	2·1	2·5	2·9	3·3
365	6·0	·3	·8	1·1	1·4	1·8	2·3	2·5	2·9	3·3
370	6·1	·4	·7	1·1	1·5	1·8	2·3	2·6	2·9	3·3
375	6·2	·4	·7	1·2	1·5	1·8	2·3	2·6	3·0	3·4
380	−6·3	+·4	·8	1·1	1·5	2·0	2·2	2·7	3·0	3·4
385	6·4	·5	·7	1·1	1·6	2·0	2·2	2·7	3·1	3·5
390	6·5	·5	·8	1·1	1·5	2·0	2·4	2·7	3·1	3·5
395	6·5	·4	·7	1·3	1·5	2·0	2·4	2·7	3·2	3·6
400	6·6	·4	·8	1·2	1·6	2·0	2·4	2·8	3·2	3·6
405	−6·7	+·4	·8	1·3	1·6	2·0	2·4	2·9	3·2	3·6
410	6·8	·5	·8	1·2	1·6	2·1	2·4	2·9	3·3	3·7
415	6·9	·5	·8	1·3	1·6	2·0	2·6	2·9	3·3	3·7
420	6·9	·4	·8	1·3	1·7	2·0	2·6	2·9	3·4	3·8
425	7·0	·4	·9	1·2	1·7	2·2	2·5	3·0	3·4	3·8
430	−7·1	+·4	·9	1·3	1·7	2·2	2·5	3·0	3·5	3·9
435	7·2	·5	·8	1·3	1·8	2·2	2·5	3·1	3·5	3·9
440	7·3	·5	·9	1·3	1·7	2·2	2·7	3·1	3·5	3·9
445	7·3	·4	·9	1·3	1·8	2·2	2·7	3·1	3·6	4·0
450	7·4	·4	·9	1·4	1·8	2·2	2·7	3·2	3·6	4·0
455	−7·5	+·4	1·0	1·3	1·9	2·2	2·7	3·2	3·7	4·1
460	7·6	·5	·9	1·4	1·8	2·3	2·7	3·3	3·7	4·1
465	7·7	·5	·9	1·4	1·9	2·3	2·8	3·3	3·7	4·1
470	7·8	·5	1·0	1·4	1·8	2·4	2·8	3·3	3·8	4·2
475	7·8	·4	1·0	1·4	1·9	2·4	2·8	3·4	3·8	4·2
480	−7·9	+·5	·9	1·4	2·0	2·4	2·8	3·4	3·9	4·3
485	8·0	·5	·9	1·5	2·0	2·4	2·8	3·5	3·9	4·3
490	8·1	·5	1·0	1·5	1·9	2·4	3·0	3·5	3·9	4·3
495	8·2	·5	1·1	1·4	2·0	2·4	3·0	3·5	4·0	4·4
500	−8·2	+·4	1·0	1·6	2·0	2·4	3·0	3·6	4·0	4·4

Interval 10, *Third Difference* c_1 (continued).

c_1	1	2	3	4	5	6	7	8	9	10
505	− 8·3	+ ·4	1·1	1·5	2·0	2·6	3·0	3·5	4·0	4·6
510	8·4	·5	1·0	1·5	2·1	2·6	3·0	3·6	4·0	4·6
515	8·5	·5	1·1	1·5	2·0	2·6	3·2	3·5	4·1	4·7
520	8·6	·6	1·0	1·5	2·1	2·6	3·2	3·6	4·1	4·7
525	8·7	·6	1·0	1·6	2·1	2·6	3·2	3·6	4·2	4·8
530	− 8·8	+ ·6	1·1	1·5	2·2	2·6	3·2	3·7	4·2	4·8
535	8·8	·5	1·1	1·6	2·1	2·7	3·2	3·7	4·3	4·9
540	8·9	·5	1·1	1·6	2·2	2·7	3·3	3·7	4·3	4·9
545	9·0	·6	1·0	1·7	2·1	2·8	3·3	3·8	4·3	4·9
550	9·1	·6	1·1	1·6	2·2	2·8	3·3	3·8	4·4	5·0
555	− 9·2	+ ·6	1·1	1·7	2·2	2·8	3·3	3·9	4·4	5·0
560	9·2	·5	1·1	1·7	2·3	2·8	3·3	3·9	4·5	5·1
565	9·3	·5	1·2	1·7	2·2	2·8	3·5	3·9	4·5	5·1
570	9·4	·6	1·1	1·7	2·3	2·8	3·5	4·0	4·5	5·1
575	9·5	·6	1·1	1·8	2·3	2·8	3·5	4·0	4·6	5·2
580	− 9·6	+ ·6	1·2	1·7	2·3	3·0	3·4	4·1	4·6	5·2
585	9·7	·7	1·1	1·7	2·4	3·0	3·4	4·1	4·7	5·3
590	9·8	·7	1·2	1·7	2·3	3·0	3·6	4·1	4·7	5·3
595	9·8	·6	1·1	1·9	2·3	3·0	3·6	4·1	4·8	5·4
600	9·9	·6	1·2	1·8	2·4	3·0	3·6	4·2	4·8	5·4
605	− 10·0	+ ·6	1·2	1·9	2·4	3·0	3·6	4·3	4·8	5·4
610	10·1	·7	1·2	1·8	2·4	3·1	3·6	4·3	4·9	5·5
615	10·2	·7	1·2	1·9	2·4	3·0	3·8	4·3	4·9	5·5
620	10·2	·6	1·2	1·9	2·5	3·0	3·8	4·3	5·0	5·6
625	10·3	·6	1·3	1·8	2·5	3·2	3·7	4·4	5·0	5·6
630	− 10·4	+ ·6	1·3	1·9	2·5	3·2	3·7	4·4	5·1	5·7
635	10·5	·7	1·2	1·9	2·6	3·2	3·7	4·5	5·1	5·7
640	10·6	·7	1·3	1·9	2·5	3·2	3·9	4·5	5·1	5·7
645	10·6	·6	1·3	1·9	2·6	3·2	3·9	4·5	5·2	5·8
650	10·7	·6	1·3	2·0	2·6	3·2	3·9	4·6	5·2	5·8
655	− 10·8	+ ·6	1·4	1·9	2·7	3·2	3·9	4·6	5·3	5·9
660	10·9	·7	1·3	2·0	2·6	3·3	3·9	4·7	5·3	5·9
665	11·0	·7	1·3	2·0	2·7	3·3	4·0	4·7	5·3	5·9
670	11·1	·7	1·4	2·0	2·6	3·4	4·0	4·7	5·4	6·0
675	11·1	·6	1·4	2·0	2·7	3·4	4·0	4·8	5·4	6·0
680	− 11·2	+ ·7	1·3	2·0	2·8	3·4	4·0	4·8	5·5	6·1
685	11·3	·7	1·3	2·1	2·8	3·4	4·0	4·9	5·5	6·1
690	11·4	·7	1·4	2·1	2·7	3·4	4·2	4·9	5·5	6·1
695	11·5	·7	1·5	2·0	2·8	3·4	4·2	4·9	5·6	6·2
700	11·5	·6	1·4	2·2	2·8	3·4	4·2	5·0	5·6	6·2
705	− 11·6	+ ·6	1·5	2·1	2·8	3·6	4·2	4·9	5·6	6·4
710	11·7	·7	1·4	2·1	2·9	3·6	4·2	5·0	5·6	6·4
715	11·8	·7	1·5	2·1	2·8	3·6	4·4	4·9	5·7	6·5
720	11·9	·8	1·4	2·1	2·9	3·6	4·4	5·0	5·7	6·5
725	12·0	·8	1·4	2·2	2·9	3·6	4·4	5·0	5·8	6·6
730	− 12·1	+ ·8	1·5	2·1	3·0	3·6	4·4	5·1	5·8	6·6
735	12·1	·7	1·5	2·2	2·9	3·7	4·4	5·1	5·9	6·7
740	12·2	·7	1·5	2·2	3·0	3·7	4·5	5·1	5·9	6·7
745	12·3	·8	1·4	2·3	2·9	3·8	4·5	5·2	5·9	6·7
750	− 12·4	+ ·8	1·5	2·2	3·0	3·8	4·5	5·2	6·0	6·8

Interval 10, *Third Difference* c_1 (continued).

c_1	1	2	3	4	5	6	7	8	9	10
755	−12·5	+ ·8	1·5	2·3	3·0	3·8	4·5	5·3	6·0	6·8
760	12·5	·7	1·5	2·3	3·1	3·8	4·5	5·3	6·1	6·9
765	12·6	·7	1·6	2·3	3·0	3·8	4·7	5·3	6·1	6·9
770	12·7	·8	1·5	2·3	3·1	3·8	4·7	5·4	6·1	6·9
775	12·8	·8	1·5	2·4	3·1	3·8	4·7	5·4	6·2	7·0
780	−12·9	+ ·8	1·6	2·3	3·1	4·0	4·6	5·5	6·2	7·0
785	13·0	·9	1·5	2·3	3·2	4·0	4·6	5·5	6·3	7·1
790	13·1	·9	1·6	2·3	3·1	4·0	4·8	5·5	6·3	7·1
795	13·1	·8	1·5	2·5	3·1	4·0	4·8	5·5	6·4	7·2
800	13·2	·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2
805	−13·3	+ ·8	1·6	2·5	3·2	4·0	4·8	5·7	6·4	7·2
810	13·4	·9	1·6	2·4	3·2	4·1	4·8	5·7	6·5	7·3
815	13·5	·9	1·6	2·5	3·2	4·0	5·0	5·7	6·5	7·3
820	13·5	·8	1·6	2·5	3·3	4·0	5·0	5·7	6·6	7·4
825	13·6	·8	1·7	2·4	3·3	4·2	4·9	5·8	6·6	7·4
830	−13·7	+ ·8	1·7	2·5	3·3	4·2	4·9	5·8	6·7	7·5
835	13·8	·9	1·6	2·5	3·4	4·2	4·9	5·9	6·7	7·5
840	13·9	·9	1·7	2·5	3·3	4·2	5·1	5·9	6·7	7·5
845	13·9	·8	1·7	2·5	3·4	4·2	5·1	5·9	6·8	7·6
850	14·0	·8	1·7	2·6	3·4	4·2	5·1	6·0	6·8	7·6
855	−14·1	+ ·8	1·8	2·5	3·5	4·2	5·1	6·0	6·9	7·7
860	14·2	·9	1·7	2·6	3·4	4·3	5·1	6·1	6·9	7·7
865	14·3	·9	1·7	2·6	3·5	4·3	5·2	6·1	6·9	7·7
870	14·4	·9	1·8	2·6	3·4	4·4	5·2	6·1	7·0	7·8
875	14·4	·8	1·8	2·6	3·5	4·4	5·2	6·2	7·0	7·8
880	−14·5	+ ·9	1·7	2·6	3·6	4·4	5·2	6·2	7·1	7·9
885	14·6	·9	1·7	2·7	3·6	4·4	5·2	6·3	7·1	7·9
890	14·7	·9	1·8	2·7	3·5	4·4	5·4	6·3	7·1	7·9
895	14·8	·9	1·9	2·6	3·6	4·4	5·4	6·3	7·2	8·0
900	14·8	·8	1·8	2·8	3·6	4·4	5·4	6·4	7·2	8·0
905	−14·9	+ ·8	1·9	2·7	3·6	4·6	5·4	6·3	7·2	8·2
910	15·0	·9	1·8	2·7	3·7	4·6	5·4	6·4	7·2	8·2
915	15·1	·9	1·9	2·7	3·6	4·6	5·6	6·3	7·3	8·3
920	15·2	1·0	1·8	2·7	3·7	4·6	5·6	6·4	7·3	8·3
925	15·3	1·0	1·8	2·8	3·7	4·6	5·6	6·4	7·4	8·4
930	−15·4	+1·0	1·9	2·7	3·8	4·6	5·6	6·5	7·4	8·4
935	15·4	·9	1·9	2·8	3·7	4·7	5·6	6·5	7·5	8·5
940	15·5	·9	1·9	2·8	3·8	4·7	5·7	6·5	7·5	8·5
945	15·6	1·0	1·8	2·9	3·7	4·8	5·7	6·6	7·5	8·5
950	15·7	1·0	1·9	2·8	3·8	4·8	5·7	6·6	7·6	8·6
955	−15·8	+1·0	1·9	2·9	3·8	4·8	5·7	6·7	7·6	8·6
960	15·8	·9	1·9	2·9	3·9	4·8	5·7	6·7	7·7	8·7
965	15·9	·9	2·0	2·9	3·8	4·8	5·9	6·7	7·7	8·7
970	16·0	1·0	1·9	2·9	3·9	4·8	5·9	6·8	7·7	8·7
975	16·1	1·0	1·9	3·0	3·9	4·8	5·9	6·8	7·8	8·8
980	−16·2	+1·0	2·0	2·9	3·9	5·0	5·8	6·9	7·8	8·8
985	16·3	1·1	1·9	2·9	4·0	5·0	5·8	6·9	7·9	8·9
990	16·4	1·1	2·0	2·9	3·9	5·0	6·0	6·9	7·9	8·9
995	16·4	1·0	1·9	3·1	3·9	5·0	6·0	6·9	8·0	9·0
1,000	−16·5	+1·0	2·0	3·0	4·0	5·0	6·0	7·0	8·0	9·0

Interval 10, *Mean Fourth Difference* (*d*).

(*d*)	1	2 10	3 9	4 8	5 7	6	(*d*)	1	2 10	3 9	4 8	5 7	6
10	+ ·1	− ·0	·1	·0	·0	·0	510	+4·0	− ·6	·9	1·0	1·0	1·0
20	·2	·1	·0	·0	·1	·0	520	4·1	·7	·9	·9	1·0	1·2
30	·2	·0	·0	·1	·1	·0	530	4·2	·8	·8	1·0	1·0	1·2
40	·3	·0	·1	·1	·1	·0	540	4·3	·8	·9	·9	1·1	1·2
50	·4	·1	·1	·0	·1	·2	550	4·3	·7	·9	1·0	1·1	1·2
60	+ ·5	− ·1	·2	·0	·1	·2	560	+4·4	− ·7	1·0	1·0	1·1	1·2
70	·5	·0	·2	·1	·1	·2	570	4·5	·8	·9	1·0	1·2	1·2
80	·6	·1	·1	·1	·2	·2	580	4·5	·7	·9	1·0	1·2	1·2
90	·7	·1	·2	·1	·2	·2	590	4·6	·7	1·0	1·1	1·2	1·2
100	·8	·2	·1	·2	·2	·2	600	4·7	·8	·9	1·2	1·2	1·2
110	+ ·9	− ·2	·2	·1	·3	·2	610	+4·8	− ·8	1·0	1·1	1·3	1·2
120	·9	·1	·2	·2	·3	·2	620	4·9	·9	1·1	1·2	1·3	1·2
130	1·0	·1	·3	·2	·3	·2	630	4·9	·7	1·1	1·2	1·2	1·4
140	1·1	·2	·2	·3	·2	·4	640	5·0	·8	1·1	1·1	1·3	1·4
150	1·2	·2	·3	·3	·2	·4	650	5·1	·8	1·1	1·3	1·2	1·4
160	+1·3	− ·3	·2	·3	·4	·2	660	+5·2	− ·9	1·0	1·3	1·3	1·4
170	1·3	·2	·2	·4	·3	·4	670	5·2	·8	1·1	1·2	1·4	1·4
180	1·4	·2	·3	·4	·3	·4	680	5·3	·9	1·0	1·2	1·4	1·4
190	1·5	·3	·2	·4	·4	·4	690	5·4	·9	1·1	1·3	1·4	1·4
200	1·6	·3	·3	·4	·4	·4	700	5·5	·9	1·1	1·4	1·4	1·4
210	+1·6	− ·2	·3	·5	·4	·4	710	+5·6	−1·0	1·1	1·3	1·4	1·6
220	1·7	·2	·4	·5	·4	·4	720	5·7	1·0	1·2	1·3	1·4	1·6
230	1·8	·3	·4	·4	·4	·6	730	5·7	·9	1·2	1·4	1·4	1·6
240	1·9	·3	·4	·5	·5	·4	740	5·8	1·0	1·1	1·4	1·5	1·6
250	2·0	·4	·4	·4	·5	·6	750	5·9	1·0	1·2	1·4	1·5	1·6
260	+2·0	− ·3	·4	·5	·5	·6	760	+5·9	− ·9	1·2	1·5	1·5	1·6
270	2·1	·3	·5	·5	·5	·6	770	6·0	·9	1·3	1·5	1·5	1·6
280	2·2	·4	·4	·5	·6	·6	780	6·1	1·0	1·2	1·5	1·6	1·6
290	2·3	·4	·5	·5	·6	·6	790	6·2	1·0	1·3	1·5	1·6	1·6
300	2·3	·3	·5	·6	·6	·6	800	6·3	1·1	1·2	1·6	1·6	1·6
310	+2·4	− ·3	·6	·6	·6	·6	810	+6·3	− ·9	1·4	1·5	1·7	1·6
320	2·5	·4	·5	·6	·7	·6	820	6·4	1·0	1·3	1·6	1·7	1·6
330	2·6	·4	·6	·6	·7	·6	830	6·5	1·0	1·4	1·6	1·6	1·8
340	2·7	·5	·5	·7	·6	·8	840	6·6	1·1	1·4	1·5	1·7	1·8
350	2·7	·4	·6	·6	·7	·8	850	6·7	1·2	1·3	1·6	1·7	1·8
360	+2·8	− ·4	·6	·7	·8	·6	860	+6·7	−1·0	1·5	1·5	1·8	1·8
370	2·9	·5	·5	·8	·7	·8	870	6·8	1·1	1·4	1·6	1·8	1·8
380	3·0	·5	·7	·6	·8	·8	880	6·9	1·1	1·5	1·6	1·8	1·8
390	3·1	·6	·6	·7	·8	·8	890	7·0	1·2	1·4	1·7	1·8	1·8
400	3·1	·5	·6	·7	·9	·8	900	7·0	1·1	1·4	1·7	1·9	1·8
410	+3·2	− ·5	·7	·7	·9	·8	910	+7·1	−1·1	1·5	1·7	1·9	1·8
420	3·3	·6	·6	·8	·9	·8	920	7·2	1·2	1·4	1·8	1·8	2·0
430	3·4	·6	·7	·8	·8	1·0	930	7·3	1·2	1·5	1·8	1·8	2·0
440	3·4	·5	·7	·9	·8	1·0	940	7·4	1·3	1·4	1·8	1·9	2·0
450	3·5	·5	·8	·8	·9	1·0	950	7·4	1·1	1·6	1·8	1·9	2·0
460	+3·6	− ·6	·7	·9	·9	1·0	960	+7·5	−1·2	1·5	1·9	1·9	2·0
470	3·7	·6	·8	·9	·9	1·0	970	7·6	1·2	1·6	1·9	1·9	2·0
480	3·8	·7	·7	·9	1·0	1·0	980	7·7	1·3	1·5	1·9	2·0	2·0
490	3·8	·5	·9	·9	1·0	1·0	990	7·8	1·3	1·6	1·9	2·0	2·0
500	+3·9	− ·6	·8	1·0	1·0	1·0	1,000	+7·9	−1·4	1·6	1·8	2·0	2·2

Central Formulæ.

Let arithmetical mean values

$$\begin{aligned}
(\mathrm{V}) &= \tfrac{1}{2}(\mathrm{V}_0+\mathrm{V}_1)\\
(b) &= \tfrac{1}{2}(b_0+b_1)\\
(d) &= \tfrac{1}{2}(d_0+d_1)\\
(f) &= \tfrac{1}{2}(f_0+f_1)\\
(h) &= \tfrac{1}{2}(h_0+h_1)
\end{aligned}$$

be inserted in the middle of the differencing, as in the following scheme:—

$$\begin{array}{c|c|c|c|c|c|c|c|c|c}
\mathrm{V}_{-4} & & & & & & & & & \\
 & a_{-4} & & & & & & & & \\
\mathrm{V}_{-3} & & b_{-3} & & & & & & & \\
 & a_{-3} & & c_{-3} & & & & & & \\
\mathrm{V}_{-2} & & b_{-2} & & d_{-2} & & & & & \\
 & a_{-2} & & c_{-2} & & e_{-2} & & & & \\
\mathrm{V}_{-1} & & b_{-1} & & d_{-1} & & f_{-1} & & & \\
 & a_{-1} & & c_{-1} & & e_{-1} & & g_{-1} & & \\
\mathrm{V}_0 & & b_0 & & d_0 & & f_0 & & h_0 & \\
(\mathrm{V}) & a_{+1} & (b) & c_{+1} & (d) & e_{+1} & (f) & g_{+1} & (h) & i_{+1} \\
\mathrm{V}_{+1} & & b_{+1} & & d_{+1} & & f_{+1} & & h_{+1} & \\
 & a_{+2} & & c_{+2} & & e_{+2} & & g_{+2} & & \\
\mathrm{V}_{+2} & & b_{+2} & & d_{+2} & & f_{+2} & & & \\
 & a_{+3} & & c_{+3} & & e_{+3} & & & & \\
\mathrm{V}_{+3} & & b_{+3} & & d_{+3} & & & & & \\
 & a_{+4} & & c_{+4} & & & & & & \\
\mathrm{V}_{+4} & & b_{+4} & & & & & & & \\
 & a_{+5} & & & & & & & & \\
\mathrm{V}_{+5} & & & & & & & & &
\end{array}$$

Then (V), a_1, (b), c_1, (d), e_1, (f), g_1, (h), will be centrally situated as regards the interval V_0 to V_1; and, by substituting $\mathrm{V}_0=(\mathrm{V})-\frac{1}{2}a_1$, $b_0=(b)-\frac{1}{2}c_1$, $d_0=(d)-\frac{1}{2}e_1$, $f_0=(f)-\frac{1}{2}g_1$, in the formula (β), it becomes

$$\begin{aligned}
\mathrm{V}_x=(\mathrm{V})+(x-\tfrac{1}{2})a_1 &+ \frac{x(x-1)}{2}\left((b)+\frac{x-\frac{1}{2}}{3}c_1\right)\\
&+\frac{x(x^2-1)(x-2)}{2.3.4}\left((d)+\frac{x-\frac{1}{2}}{5}e_1\right)\\
&+\frac{x(x^2-1)(x^2-4)(x-3)}{2\ldots 6}\left((f)+\frac{x-\frac{1}{2}}{7}g_1\right)\\
&+\frac{x(x^2-1)(x^2-4)(x^2-9)(x-4)}{2\ldots 8}\Big((h)+\ldots\Big)\,.\,.\,(\gamma).
\end{aligned}$$

It is now proposed to determine the central differences of the set of interpolated quantities from V_0 to V_1, in terms of the primitive central values (V), a_1, (b), c_1, &c.

In the first place we shall find general values for each order of interpolated differences.

Let n denote the number of subdivisions of the interval. Then,

in order that the formula (γ) may represent the interpolated values when $x=1, 2, 3$, &c., substitute $\frac{x}{n}$ for x, and it becomes

$$V'_x=(V)-(\tfrac{1}{2}n-x)\frac{a_1}{n}-\frac{x(n-x)}{2}\cdot\frac{(b)}{n^2}+\frac{x(n-x)(\frac{1}{2}n-x)}{2.3}\cdot\frac{c_1}{n^3}$$
$$+\frac{x(n^2-x^2)(2n-x)}{2.3.4}\cdot\frac{(d)}{n^4}-\frac{x(n^2-x^2)(2n-x)(\frac{1}{2}n-x)}{2\ldots5}\cdot\frac{e_1}{n^5}$$
$$\frac{x(n^2-x^2)(4n^2-x^2)(3n-x)}{2\ldots6}\cdot\frac{(f)}{n^6}+\frac{x(n^2-x^2)(4n^2-x^2)(3n-x)(\frac{1}{2}n-x)}{2\ldots7}\cdot\frac{g_1}{n^7}$$
$$+\frac{x(n^2-x^2)(4n^2-x^2)(9n^2-x^2)(4n-x)}{2\ldots8}\cdot\frac{(h)}{n^8}.$$

To abbreviate and simplify the expressions, assume

$$u=\tfrac{1}{2}n-x$$
$$w=\tfrac{1}{2}(n+1)-x;$$

then, following the scheme of differencing as indicated, and making the requisite substitutions and reductions, we get the following results:—

$$V'_x=(V)-u\frac{a_1}{n}+\left(u^2-\frac{n^2}{4}\right)\frac{(b)}{2n^2}-u\left(u^2-\frac{n^2}{4}\right)\frac{c_1}{2.3n^3}$$
$$+\left(u^2-\frac{n^2}{4}\right)\left(u^2-\frac{9n^2}{4}\right)\frac{(d)}{2.3.4n^4}-u\left(u^2-\frac{n^2}{4}\right)\left(u^2-\frac{9n^2}{4}\right)\frac{e_1}{2..5n^5}$$
$$+\left(u^2-\frac{n^2}{4}\right)\left(u^2-\frac{9n^2}{4}\right)\left(u^2-\frac{25n^2}{4}\right)\frac{(f)}{2\ldots6n^6}$$
$$-u\left(u^2-\frac{n^2}{4}\right)\left(u^2-\frac{9n^2}{4}\right)\left(u^2-\frac{25n^2}{4}\right)\frac{g_1}{2\ldots7n^7}$$
$$+\left(u^2-\frac{n^2}{4}\right)\left(u^2-\frac{9n^2}{4}\right)\left(u^2-\frac{25n^2}{4}\right)\left(u^2-\frac{49n^2}{4}\right)\frac{(h)}{2\ldots8n^8};$$

$$a'_x=V'_x-V'_{x-1}$$
$$=\frac{a_1}{n}-w\frac{(b)}{n^2}+\left(w^2-\frac{n^2-1}{12}\right)\frac{c_1}{2n^3}-w\left(w^2-\frac{5n^2-1}{4}\right)\frac{(d)}{2.3n^4}$$
$$+\left(w^4-\frac{3n^2-1}{2}w^2+\frac{9n^4-10n^2+1}{80}\right)\frac{e_1}{2.3.4n^5}$$
$$-w\left(w^4-\frac{5(7n^2-1)}{6}w^2+\frac{259n^4-70n^2+3}{48}\right)\frac{(f)}{2\ldots5n^6}$$
$$+\left(w^6-\frac{5(5n^2-1)}{4}w^4+\frac{111n^4-50n^2+3}{16}w^2\right.$$
$$\left.-\frac{225n^6-259n^4+35n^2-1}{7.64}\right)\frac{g_1}{2\ldots6n^7}$$

$$-w\left(w^6-\frac{7(9n^2-1)}{4}w^4+\frac{7(141n^4-30n^2+1)}{16}w^2\right.$$
$$\left.-\frac{3229n^6-987n^4+63n^2-1}{64}\right)\frac{(h)}{2\ldots 7n^8};$$

$$b'_x=a'_{x+1}-a'_x$$
$$=\frac{(b)}{n^2}-u\frac{c_1}{n^3}+\left(u^2-\frac{5n^2-2}{12}\right)\frac{(d)}{2n^4}-u\left(u^2-\frac{3n^2-2}{4}\right)\frac{e_1}{2.3n^5}$$
$$+\left(u^4-\frac{7n^2-2}{2}u^2+\frac{259n^4-140n^2+16}{240}\right)\frac{(f)}{2.3.4n^6}$$
$$-u\left(u^4-\frac{5(5n^2-2)}{6}u^2+\frac{111n^4-100n^2+16}{48}\right)\frac{g_1}{2\ldots 5n^7}$$
$$+\left(u^6-\frac{5(9n^2-2)}{4}u^4+\frac{423n^4-180n^2+16}{16}u^2\right.$$
$$\left.-\frac{3229n^6-1974n^4+336n^2-16}{7.64}\right)\frac{(h)}{2\ldots 6n^8};$$

$$c'_x=b'_x-b'_{x-1}$$
$$=\frac{c_1}{n^3}-w\frac{(d)}{n^4}+\left\{w^2-\frac{n^2-1}{4}\right\}\frac{e_1}{2n^5}-w\left\{w^2-\frac{7n^2-3}{4}\right\}\frac{(f)}{2.3n^6}$$
$$+\left\{w^4\quad\frac{5n^2-3}{2}w^2+\frac{37n^4-50n^2+13}{80}\right\}\frac{g_1}{2.3.4n^7}$$
$$-w\left\{w^4-\frac{5(3n^2-1)}{2}w^2+\frac{141n^4-90n^2+13}{16}\right\}\frac{(h)}{2\ldots 5n^8};$$

$$d'_x=c'_{x+1}-c'_x$$
$$=\frac{(d)}{n^4}-u\frac{e_1}{n^5}+\left\{u^2-\frac{7n^2-4}{12}\right\}\frac{(f)}{2n^6}-u\left\{u^2-\frac{5n^2-4}{4}\right\}\frac{g_1}{2.3n^7}$$
$$+\left\{u^4-\frac{9n^2-4}{2}u^2+\frac{141n^4-120n^2+24}{80}\right\}\frac{(h)}{2.3.4n^8};$$

$$e'_x=d'_x-d'_{x-1}$$
$$=\frac{e_1}{n^5}-w\frac{(f)}{n^6}+\left\{w^2-\frac{5(n^2-1)}{12}\right\}\frac{g_1}{2n^7}-w\left\{w^2-\frac{9n^2-5}{4}\right\}\frac{(h)}{2.3n^8};$$

$$f'_x=e'_{x+1}-e'_x$$
$$=\frac{(f)}{n^6}-u\frac{g_1}{n^7}+\left\{u^2-\frac{3n^2-2}{4}\right\}\frac{(h)}{2n^8};$$

$$g'_x=f'_x-f'_{x-1}$$
$$=\frac{g_1}{n^7}-w\frac{(h)}{n^8};$$

$$h'_x=\frac{(h)}{n^8}.$$

Now the disposition of the central values will be different, according as n is even or odd, and the two cases must therefore be taken separately.

When n is even, the central values V', b', d', f', h', will obviously be determined by making $x=\frac{1}{2}n$, or $u=0$; and the mean central values (a'), (c'), (e'), (g'), will be obtained by taking half the sum of the values when $x=\frac{1}{2}n$ and $\frac{1}{2}n+1$, or when $w=\pm\frac{1}{2}$. Hence, making these calculations, and, to simplify the notation, removing the epoch to the middle of the interpolated interval, we get the following values:—

When n is even,

$$V'_0=(V)-\frac{1}{2}\frac{(b)}{4}+\frac{1.3}{2.4}\frac{(d)}{4^2}-\frac{1.3.5}{2.4.6}\cdot\frac{(f)}{4^3}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{(h)}{4^4}$$

$$(a'_0)=\frac{a_1}{n}-\frac{n^2-4}{2.3.4}\cdot\frac{c_1}{n^3}+\frac{(n^2-4)(9n^2-4)}{2\ldots5.4^2}\cdot\frac{e_1}{n^5}$$
$$-\frac{(n^2-4)(9n^2-4)(25n^2-4)}{2\ldots7.4^3}\cdot\frac{g_1}{n^7}$$

$$b'_0=\frac{(b)}{n^2}-\frac{5n^2-2}{2.3.4}\cdot\frac{(d)}{n^4}+\frac{259n^4-140n^2+16}{2\ldots6.8}\cdot\frac{(f)}{n^6}$$
$$-\frac{3229n^6-1974n^4+336n^2-16}{2\ldots8.8}\cdot\frac{(h)}{n^8}$$

$$(c'_0)=\frac{c_1}{n^3}-\frac{n^2-2}{2.4}\cdot\frac{e_1}{n^5}+\frac{37n^4-100n^2+48}{2\ldots5.16}\cdot\frac{g_1}{n^7}$$

$$d'_0=\frac{(d)}{n^4}-\frac{7n^2-4}{2.3.4}\cdot\frac{(f)}{n^6}+\frac{47n^4-40n^2+8}{2.4.5.16}\cdot\frac{(h)}{n^8}$$

$$(e'_0)=\frac{e_1}{n^5}-\frac{5n^2-8}{2.3.4}\cdot\frac{g_1}{n^7}$$

$$f'_0=\frac{(f)}{n^6}-\frac{3n^2-2}{2.4}\cdot\frac{(h)}{n^8}$$

$$(g'_0)=\frac{g_1}{n^7}$$

$$h'_0=\frac{(h)}{n^8}\quad.\quad.\quad.\quad.\quad.\quad(q)^*$$

Again, when n is odd, the values of a', c', e', g', will be determined by making $x=\frac{1}{2}(n+1)$, or $w=0$; and the mean values (V'), (b'), (d'), (f'), (h'), will be half the sum of the values when $x=\frac{1}{2}(n-1)$ and $\frac{1}{2}(n+1)$, or when $u=\pm\frac{1}{2}$. Proceeding thus, we obtain—

When n is odd,

$$(V'_0)=(V)-\frac{n^2-1}{2.4}\cdot\frac{(b)}{n^2}+\frac{(n^2-1)(9n^2-1)}{2.3.4.4^2}\cdot\frac{(d)}{n^4}$$
$$-\frac{(n^2-1)(9n^2-1)(25n^2-1)}{2\ldots6.4^3}\cdot\frac{(f)}{n^6}+\text{\&c.}$$

* In the formula (p), page 67, which have reference to the original epoch V_0, n may be either even or odd.

$$a'_0=\frac{a_1}{n}-\frac{n^2-1}{2.3.4}\cdot\frac{c_1}{n^3}+\frac{(n^2-1)(9n^2-1)}{2\ldots5.4}\cdot\frac{e_1}{n^5}$$
$$-\frac{(n^2-1)(9n^2-1)(25n^2-1)}{2\ldots7.4^3}\cdot\frac{g_1}{n^7}$$
$$(b'_0)=\frac{(b)}{n^2}-\frac{5(n^2-1)}{2.3.4}\cdot\frac{(d)}{n^4}+\frac{7(n^2-1)(37n^2-13)}{2\ldots6.8}\frac{(f)}{n^6}$$
$$-\frac{(n^2-1)(3229n^4-1706n^2+205)}{2\ldots8.8}\cdot\frac{(h)}{n^8}$$
$$c'_0=\frac{c_1}{n^3}-\frac{n^2-1}{2.4}\cdot\frac{e_1}{n^5}+\frac{(n^2-1)(37n^2-13)}{2\ldots5.16}\cdot\frac{g_1}{n^7}$$
$$(d'_0)=\frac{(d)}{n^4}-\frac{7(n^2-1)}{2.3.4}\cdot\frac{(f)}{n^6}+\frac{(n^2-1)(47n^2-23)}{2.4.5.16}\cdot\frac{(h)}{n^8}$$
$$e'_0=\frac{e_1}{n^5}-\frac{5(n^2-1)}{2.3.4}\cdot\frac{g_1}{n^7}$$
$$(f'_0)=\frac{(f)}{n^6}-\frac{3(n^2-1)}{2.4}\cdot\frac{(h)}{n^8}$$
$$g'_0=\frac{g_1}{n^7}$$
$$(h'_0)=\frac{(h)}{n^8}\quad\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\quad(r)^*$$

When restricted to four orders of differences these formulæ become comparatively simple. Thus—

When n is even,

$$V'_0=(V)-\frac{1}{2}\frac{(b)}{4}+\frac{1.3}{2.4}\frac{(d)}{4^2}$$
$$(a'_0)=\frac{a_1}{n}-\frac{n^2-4}{2.3.4}\cdot\frac{c_1}{n^3}$$
$$b'_0=\frac{(b)}{n^2}-\frac{5n^2-2}{2.3.4}\cdot\frac{(d)}{n^4}$$
$$(c'_0)=\frac{c_1}{n^3}$$
$$d'_0=\frac{(d)}{n^4}\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;(q')$$

When n is odd,

$$(V'_0)=(V)-\frac{n^2-1}{2.4}\frac{(b)}{n^2}+\frac{(n^2-1).(9n^2-1)}{2.3.4^3}\frac{(d)}{n^4}$$
$$a'_0=\frac{a_1}{n}-\frac{n^2-1}{2.3.4}\cdot\frac{c_1}{n^3}$$

* See Note, page 86.

$$(b'_0)=\frac{(b)}{n^2}-\frac{5(n^2-1)}{2.3.4}\cdot\frac{(d)}{n^4}$$

$$c'_0=\frac{c_1}{n^3}$$

$$(d'_0)=\frac{(d)}{n^4}\;.\;.\;.\;.\;.\;(r').$$

By means of the inverse operations of differencing, extending both upwards and downwards, according to the scheme, all the interpolated values may be readily found from the central quantities given by (q) or (r).

Thus, let the differencing be carried to the fourth order, and dispense with the accentuation; then, when the number (n) of intervals is even, the scheme as determined from central values V, (a), b, (c), d, will be as follows:—

	&c.				
	$V-3(a)+\frac{9}{2}b-4(c)+3d$	$(a)-\frac{5}{2}b+3(c)-\frac{5}{2}d$	$b-3(c)+\frac{9}{2}d$	$(c)-\frac{5}{2}d$	
	$V-2(a)+2b-(c)+\frac{1}{2}d$	$(a)-\frac{3}{2}b+(c)-\frac{1}{2}d$	$b-2(c)+2d$	$(c)-\frac{3}{2}d$	
	$V-(a)+\frac{1}{2}b$	$(a)-\frac{1}{2}b$	$b-(c)+\frac{1}{2}d$	$(c)-\frac{1}{2}d$	
....	V	(a)	b	(c)	d
	$V+(a)+\frac{1}{2}b$	$(a)+\frac{1}{2}b$	$b+(c)+\frac{1}{2}d$	$(c)+\frac{1}{2}d$	
	$V+2(a)+2b+(c)+\frac{1}{2}d$	$(a)+\frac{3}{2}b+(c)+\frac{1}{2}d$	$b+2(c)+2d$	$(c)+\frac{3}{2}d$	
	$V+3(a)+\frac{9}{2}b+4(c)+3d$	$(a)+\frac{5}{2}b+3(c)+\frac{5}{2}d$	$b+3(c)+\frac{9}{2}d$	$(c)+\frac{5}{2}d$	
	&c.				

And when the number (n) of intervals is odd, the scheme similarly derived from central values (V), a, (b), c, (d), will appear thus:—

	&c.				
	$(V)-\frac{5}{2}a+3(b)-\frac{5}{2}c+(d)$	$a-2(b)+2c-(d)$	$(b)-\frac{5}{2}c+3(d)$	$c-2(d)$	
	$(V)-\frac{3}{2}a+(b)-\frac{1}{2}c$ . . .	$a-(b)+\frac{1}{2}c$ . . .	$(b)-\frac{3}{2}c+(d)$	$c-(d)$	
	$(V)-\frac{1}{2}a$		$(b)-\frac{1}{2}c$		
....	(V)	a	(b)	c . . .	(d)
	$(V)+\frac{1}{2}a$		$(b)+\frac{1}{2}c$		
	$(V)+\frac{3}{2}a+(b)+\frac{1}{2}c$ . . .	$a+(b)+\frac{1}{2}c$ . . .	$(b)+\frac{3}{2}c+(d)$	$c+(d)$	
	$(V)+\frac{5}{2}a+3(b)+\frac{5}{2}c+(d)$	$a+2(b)+2c+(d)$	$(b)+\frac{5}{2}c+3(d)$	$b+2(d)$	
	&c.				

(*To be continued.*)